

Low Energy Constants from High Energy Theorems

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ABSTRACT

New constraints on resonance saturation in chiral perturbation theory are investigated. These constraints arise because each consistent saturation scheme must map to a representation of the full QCD chiral symmetry group. The low-energy constants of chiral perturbation theory are then related by a set of mixing angles. It is shown that vector meson dominance is a consequence of the fact that nature has chosen the lowest-dimensional nontrivial chiral representation. It is further shown that chiral symmetry places an upper bound on the mass of the lightest scalar in the hadron spectrum.

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1. Introduction

The experimentally determined low-energy constants of chiral perturbation theory are in excellent agreement with low-energy constants determined by resonance saturation [1]. This is no surprise: the low-energy constants of chiral perturbation theory represent the effect of resonances which have been integrated out of the low-energy effective theory. What is surprising is that only a few low-lying resonances account for all of the strength of the chiral perturbation theory parameters. One might think it natural that the lowest-lying states dominate. However, there is no separation of scales in the spectrum which would indicate that only a given set of low-lying resonances should dominate over all others at low energies. Interestingly, large- N arguments suggest that an infinite number of resonances contribute with more or less equal strength, an expectation which is realized in string-like models of $\pi - \pi$ scattering [2].

Various theoretical constraints on resonance saturation have been investigated [3]. Foremost among these constraints are the spectral function sum rules [4], which are chiral symmetry constraints on products of two QCD currents [5]. In a recent paper it was shown that there are analogous sum rules for products of three and four QCD currents [6]. These sum rules also imply constraints on resonance saturation. The purpose of this paper is to consider all of the relevant constraints at once in the case of $\pi - \pi$ scattering. There is then a simple chiral symmetry interpretation of these constraints: all particles in a given saturation scheme are in a representation of the full QCD chiral symmetry group together with the pions [6]. This interpretation clarifies resonance saturation and sheds light on the ancient notion of vector meson dominance.

The reader may worry that because chiral symmetry is spontaneously broken there is little sense in classifying states in the low-energy theory using chiral symmetry. This worry is unfounded. A helpful way to think is in terms of the operator product expansion, where it is straightforward to prove that coefficient functions transform with respect to the full global symmetry group of QCD, in spite of the fact that this symmetry is spontaneously broken [5]. It is in fact precisely this property of the operator product expansion which leads to the sum rules which we study. The reader is referred to Ref. 6 for details.

In section 2 we write down the most general $SU(2)_L \times SU(2)_R$ invariant chiral lagrangian to order p^4 in the chiral limit. We then consider the general theory of resonance saturation in section 3 and demonstrate the remarkable success of the vector meson dominance picture. In section 4 we write down the complete set of chiral sum rules relevant to $\pi - \pi$ scattering. We then consider the simplest saturation schemes in section 5. We also derive an upper bound on the mass of the lowest-lying scalar and consider explicit chiral symmetry breaking effects. We conclude in section 6.

2. Low-Energy Constants

Consider the most general low-energy lagrangian consistent with the symmetry breaking pattern $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$. This low-energy lagrangian accommodates all underlying theories that share this pattern of symmetry breaking. Well known technology tells how to build the most general lagrangian involving pions consistent with the relevant pattern of symmetry breaking [7]. We introduce a field U that transforms linearly with respect to $SU(2)_L \times SU(2)_R$: $U \rightarrow LUR^\dagger$ where L, R is an element of $SU(2)_{L,R}$. A convenient parameterization of U is

$$U = \exp \frac{i\pi_a \tau_a}{F_\pi} \quad (1)$$

where τ_a are the Pauli matrices and π_a is the canonical pion field. The effective lagrangian describing the interactions of the pion at low energies can be expressed as $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \dots$ where the subscripts refer to the number of derivatives. The leading operator in the chiral limit is

$$\mathcal{L}_2 = \frac{1}{4} F_\pi^2 \text{tr} (D_\mu U D^\mu U^\dagger). \quad (2)$$

Here we use the covariant derivative $D_\mu = \partial_\mu - ir_\mu U + iUl_\mu$ where r_μ and l_μ are external fields with associated non-abelian field strengths $F_R^{\mu\nu}$ and $F_L^{\mu\nu}$, respectively.

At order p^4 there are four invariant operators in the chiral limit [1]

$$\begin{aligned} \mathcal{L}_4 = & \frac{1}{4} l_1 \text{tr} (D_\mu U D^\mu U^\dagger)^2 + \frac{1}{4} l_2 \text{tr} (D_\mu U D_\nu U^\dagger) \text{tr} (D^\mu U D^\nu U^\dagger) \\ & + l_5 \text{tr} (F_{\mu\nu L} U F_R^{\mu\nu} U^\dagger) - i \frac{1}{2} l_6 \text{tr} (F_L^{\mu\nu} D_\mu U D_\nu U^\dagger + F_R^{\mu\nu} D_\mu U^\dagger D_\nu U) \end{aligned} \quad (3)$$

where we have taken into account the coupling of the pions to the external fields. The renormalization scale-dependent parameters l_1 , l_2 , l_5 and l_6 are undetermined and independent insofar as the pattern of symmetry breaking is concerned. In the large- N limit the continuum vanishes and the l 's—which are of order N —are determined by sums of infinite numbers of narrow resonances [8].

3. Resonance Saturation

3.1 Adding Matter

It is straightforward to couple massive states with any quantum numbers to the pions in a chirally invariant way [3,7]. There are many ways of introducing resonance fields. The basic rules of quantum field theory ensure that all ways contribute the same physics at low energies [9].

The axialvector couplings of the resonances to the pions are defined by

$$\begin{aligned}
\langle \pi_b | Q_a^5 | V_c \rangle_i &= -i \epsilon_{abc} G_{V_i \pi} / F_\pi \\
\langle \pi_b | Q_a^5 | S \rangle_i &= -i \delta_{ab} G_{S_i \pi} / F_\pi
\end{aligned} \tag{4}$$

where Q_a^5 are the QCD axial charges and V and S represent states with $1^+(J^{--})$ (J odd) and $0^+(J^{++})$ (J even), respectively (here we use the standard notation: $I^G(J^{PC})$). These mesons are, respectively, the ρ_J 's with J odd and the f_J 's with J even in the particle data tables.

The couplings to the vector and axialvector currents are given by:

$$\begin{aligned}
\langle 0 | A_{a\mu} | \pi_b \rangle &= \delta_{ab} F_\pi p_\mu \\
\langle 0 | A_{a\mu} | A_b \rangle_i^{(\lambda)} &= \delta_{ab} F_{A_i} M_{A_i} \epsilon_\mu^{(\lambda)} \\
\langle 0 | V_{a\mu} | V_b \rangle_i^{(\lambda)} &= \delta_{ab} F_{V_i} M_{V_i} \epsilon_\mu^{(\lambda)}
\end{aligned} \tag{5}$$

where $V_{a\mu}$ and $A_{a\mu}$ are the (conserved) QCD vector and axialvector currents and $\epsilon_\mu^{(\lambda)}$ is the vector meson polarization vector. Of course $F_{V_i} \neq 0$ only if $J = 1$. Here A represents states with $1^-(1^{++})$. These mesons are the a_1 's in the particle data tables.

3.2 Vector Meson Dominance

Consider saturating the low-energy constants with a single V state, $\rho(770)$, and a single A state, $a_1(1260)$. We then have [1,3]

$$\begin{aligned}
l_1 &= -\frac{G_{\rho\pi}^2}{M_\rho^2} & l_2 &= \frac{G_{\rho\pi}^2}{M_\rho^2} \\
l_5 &= -\frac{F_\rho^2}{4M_\rho^2} + \frac{F_{a_1}^2}{4M_{a_1}^2} & l_6 &= -\frac{F_\rho G_{\rho\pi}}{M_\rho^2}.
\end{aligned} \tag{6}$$

Experiment gives

$$\frac{F_\pi^2}{G_{\rho\pi}^2} \simeq 1.9 \qquad \frac{F_\rho^2}{F_\pi^2} \simeq 2.7 \tag{7}$$

which are extracted from the decays $\rho \rightarrow \pi\pi$ and $\rho^0 \rightarrow e^+e^-$ [10]. The experimental situation can then be roughly summarized through the relations:

$$F_\pi = \frac{1}{\sqrt{2}} F_\rho \qquad G_{\rho\pi} = \frac{1}{\sqrt{2}} F_\pi. \tag{8}$$

Equivalently we can determine $\sqrt{2} G_{\rho\pi} = F_\pi$ from $\rho \rightarrow \pi\pi$ and then use the sum rule $G_{\rho\pi} F_\rho = F_\pi^2$ [3]. The relations

$$F_\pi = F_{a_1} \qquad M_\rho = \frac{1}{\sqrt{2}} M_{a_1} \tag{9}$$

i	$l_i^{exp} \times 10^{-3}$	Source	$l_i^{th} \times 10^{-3}$
1	-5.4 ± 2.5	$K_{e4}; \pi\pi \rightarrow \pi\pi$	-7.3
2	5.4 ± 1.2	$K_{e4}; \pi\pi \rightarrow \pi\pi$	7.3
5	-5.5 ± 0.7	$\pi \rightarrow e\nu\gamma$	-5.5
6	-13.7 ± 1.4	$\langle r^2 \rangle_V^\pi$	-14.6

Table 1: The naive VMD scheme. Coefficients at one loop order in chiral perturbation theory taken from Ref. 7, evaluated at $\mu = M_\rho$. The theoretical predictions are taken from Eq. (10)

then follow directly from spectral function sum rules. The various sum rules will be discussed in detail below. With these values of the resonance parameters the low-energy constants are related by:

$$-2l_1 = 2l_2 = -\frac{8}{3}l_5 = -l_6 = \frac{F_\pi^2}{M_\rho^2} \equiv \bar{l} \quad (10)$$

and compared to experiment in Table 1. The agreement is rather striking for such a simple saturation scheme. This is the modern version of vector meson dominance (VMD).

4. High-energy Theorems at Large- N

The saturation scheme of section 3 demonstrates the phenomenological success of the resonance saturation procedure when combined with VMD. But is this scheme consistent with QCD? Generally, one may wonder whether a saturation scenario with arbitrary particle content and arbitrary masses and couplings is consistent with what we know about QCD. This is *not* the case as there are important chiral symmetry constraints which must be satisfied.

The chiral sum rules, or high-energy theorems, which must be satisfied (in the chiral limit) are:

$$\sum_V F_V^2 - \sum_A F_A^2 = F_\pi^2 \quad (11a)$$

$$\sum_V F_V^2 M_V^2 - \sum_A F_A^2 M_A^2 = 0 \quad (11b)$$

$$\sum_V F_V G_{V\pi} = F_\pi^2 \quad (11c)$$

$$\sum_V G_{V\pi}^2 + \sum_S G_{S\pi}^2 = F_\pi^2 \quad (11d)$$

$$\sum_V G_{V\pi}^2 M_V^2 - \sum_S G_{S\pi}^2 M_S^2 = 0. \quad (11e)$$

These sum rules contain the *totality* of the constraints which the full chiral symmetry of QCD places on the low-energy constants of chiral perturbation theory in the large- N limit [6]. The first two sum rules are familiar as the spectral function sum rules at large- N [11]. The other three sum rules are known to follow from assumptions of unsubtracted dispersion relations for the pion vector form factor [3,12] and the $I_t = 1$ and $I_t = 2$ $\pi - \pi$ scattering amplitudes [13], respectively. In Ref. 6 it is shown that these five sum rules are exact in large- N QCD and follow directly from $SU(2) \times SU(2)$ symmetry.

5. Finite Dimensional Saturation Schemes

The chiral sum rules of Eq. (11) are saturated by an infinite number of states in the large- N limit. In order to connect with experiment we must consider saturation schemes with a finite number of states. Chiral symmetry is respected provided that the sum rules are satisfied.

5.1 The Trivial Scheme

The simplest scheme contains π and a single V state, the $\rho(770)$. This corresponds to the six dimensional representation $(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$. The solution to the sum rules is

$$F_\pi = F_\rho \quad M_\rho = 0 \quad G_{\rho\pi} = F_\pi. \quad (12)$$

Of course, in this case, there is no mass splitting and chiral perturbation theory can be consistent only if ρ is kept as an explicit degree of freedom, not a particularly interesting scenario.

Notice that the sigma model scenario —containing π and a single S state— which corresponds to the four dimensional $(\mathbf{2}, \mathbf{2})$ representation, cannot satisfy all of the sum rules and therefore is inconsistent with QCD, not a surprising result. Of course Eq. (11d) is satisfied for *any* representation involving π and gives $G_{S\pi} = F_\pi$, which is, as expected, the (tree-level) sigma model value.

5.2 The Simplest Nontrivial Scheme

The reader might have noticed that the VMD scheme of section 3 is not consistent with the chiral sum rules. This is because that scheme involves three isovectors, π , ρ and a_1 , which do not fit into an $SU(2) \times SU(2)$ representation. Three isovectors contain nine degrees of freedom. The only relevant representations of $SU(2) \times SU(2)$ that contain isovectors are $(\mathbf{2}, \mathbf{2})$ and $(\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{1})$, which are dimension four and six, respectively. There is simply no way to add four and six to give nine. Therefore, consistency with chiral symmetry requires that the saturation scheme include at least one additional isoscalar or

one additional isovector. Hence it is clear that the simplest nontrivial saturation scheme (i.e. with mass splittings) must contain π , $\rho(770)$, $a_1(1260)$ and one S , which we take as $f_0(400-1200)$ [14]. This corresponds to the ten dimensional representation $(\mathbf{2}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$. Since the representation is reducible there is a mixing angle. The solution of the sum rules is given in terms of the mixing angle ϕ :

$$F_\pi = F_\rho \sin \phi \quad F_{a_1} = F_\rho \cos \phi \quad (13a)$$

$$M_\rho = M_{a_1} \cos \phi \quad M_{f_0} = M_\rho \tan \phi \quad (13b)$$

$$G_{\rho\pi} = F_\pi \sin \phi \quad G_{f_0\pi} = F_\pi \cos \phi. \quad (13c)$$

It is easy to verify that the sum rules are satisfied by Eq. (13):

$$F_{a_1}^2 + F_\pi^2 = F_\rho^2 \quad (14a)$$

$$M_\rho^2 F_\rho^2 - M_{a_1}^2 F_{a_1}^2 = 0 \quad (14b)$$

$$G_{\rho\pi} F_\rho = F_\pi^2 \quad (14c)$$

$$G_{\rho\pi}^2 + G_{f_0\pi}^2 = F_\pi^2 \quad (14d)$$

$$M_\rho^2 G_{\rho\pi}^2 - M_{f_0}^2 G_{f_0\pi}^2 = 0. \quad (14e)$$

Saturating the chiral perturbation theory parameters with this particle content gives [1,3]

$$\begin{aligned} l_1 &= -\frac{G_{\rho\pi}^2}{M_\rho^2} + \frac{G_{f_0\pi}^2}{2M_{f_0}^2} & l_2 &= \frac{G_{\rho\pi}^2}{M_\rho^2} \\ l_5 &= -\frac{F_\rho^2}{4M_\rho^2} + \frac{F_{a_1}^2}{4M_{a_1}^2} & l_6 &= -\frac{F_\rho G_{\rho\pi}}{M_\rho^2}. \end{aligned} \quad (15)$$

The solution of the sum rules then yields

$$\begin{aligned} l_1 &= -\frac{1}{2}\bar{l}(\sin^2 \phi - \csc^2 \phi + 2) & l_2 &= \bar{l} \sin^2 \phi \\ l_5 &= -\frac{1}{4}\bar{l}(2 - \sin^2 \phi) & l_6 &= -\bar{l}. \end{aligned} \quad (16)$$

Now, we can fit the mixing angle to a datum. Say we determine $\sqrt{2}G_{\rho\pi} = F_\pi$ from $\rho \rightarrow \pi\pi$, which implies $\phi \sim 45^\circ$. We then have

$$F_\pi = \frac{1}{\sqrt{2}}F_\rho \quad G_{\rho\pi} = \frac{1}{\sqrt{2}}F_\pi \quad G_{f_0\pi} = \frac{1}{\sqrt{2}}F_\pi \quad (17a)$$

$$F_\pi = F_{a_1} \quad M_\rho = \frac{1}{\sqrt{2}}M_{a_1} \quad M_\rho = M_{f_0}. \quad (17b)$$

Note that only a single datum has been used. Chiral symmetry then predicts all in terms of F_π and M_ρ . In particular it is now clear that the mysterious factors of $\sqrt{2}$ are related

	TH (a,b)	EXP		TH (a,b)	EXP
F_ρ	132, 127	153 ± 4	$G_{f_0\pi}$	66, 63	$57 - 74$
F_{a_1}	93, 87	122 ± 23	M_{a_1}	1089, 1129	1230 ± 40
$G_{\rho\pi}$	66, <i>fit</i>	68 ± 1	M_{f_0}	770, 826	$400 - 1200$

Table 2: Coupling and mass predictions taken from Eq. (13) —with (a) $\phi = 45^\circ$ and (b) $\phi = 47^\circ$ (fit to $G_{\rho\pi}$)— compared to experiment [10]. We have also used $M_\rho = 770$ and $F_\pi = 93$. All numbers are in MeV.

by chiral symmetry. These relations are compared to experiment directly in Table 2. This set of relations is to be compared to Eq. (8) and Eq. (9).

Here the low-energy constants of chiral perturbation theory are related through

$$-4l_1 = 2l_2 = -\frac{8}{3}l_5 = -l_6 = \bar{l} \quad (18)$$

and are compared to experiment and to a Roy equation analysis [15] in Table 3. The sole difference with the VMD scenario of section 3 is in l_1 . Evidently both scenarios are compatible with data. We reiterate that in the method advocated here, all resonances which contribute to the low-energy constants are in a common chiral multiplet, while in Ref. 3 the resonances are essentially decoupled from one another. It would be interesting to reconsider the full three-flavor analysis of Ref. 3 using $SU(3) \times SU(3)$.

5.3 Is a Light Scalar Necessary?

Given the debatable status of the lowest-lying scalars it is interesting to see whether it is possible to develop a realistic scheme in which the lightest scalar mass is pushed up. We will see that this is difficult to achieve unless some of the successes of the VMD picture are sacrificed.

Assume that there is a single V state, $\rho(770)$, and a single A state, $a_1(1260)$, and any number of S states, and further assume that $\phi = 45^\circ$. The sum rules then imply

$$F_\pi = F_{a_1} \quad (19a)$$

$$\sqrt{2}M_\rho = M_{a_1} \quad (19b)$$

$$G_{\rho\pi} = \frac{1}{\sqrt{2}}F_\pi \quad (19c)$$

$$\sum_S G_{S\pi}^2 = \frac{1}{2}F_\pi^2 \quad (19d)$$

$$\sum_S G_{S\pi}^2 M_S^2 = \frac{1}{2}F_\pi^2 M_\rho^2. \quad (19e)$$

i	$l_i^{exp} \times 10^{-3}$	Source	$l_i^{th} \times 10^{-3}$	$l_i^{roy} \times 10^{-3}$
1	-5.4 ± 2.5	$K_{e4}; \pi\pi \rightarrow \pi\pi$	-3.7	-5.4 ± 0.2
2	5.4 ± 1.2	$K_{e4}; \pi\pi \rightarrow \pi\pi$	7.3	~ 3.4
5	-5.5 ± 0.7	$\pi \rightarrow e\nu\gamma$	-5.5	–
6	-13.7 ± 1.4	$\langle r^2 \rangle_V^\pi$	-14.6	–

Table 3: The consistent VMD scheme. Coefficients at one loop order in chiral perturbation theory taken from Ref. 7, evaluated at $\mu = M_\rho$. The theoretical predictions are taken from Eq. (18). Also included in the last column are Roy equation determinations of l_1 and l_2 [15].

Multiplying Eq. (19d) by M_ρ^2 and subtracting Eq. (19e) gives

$$\sum_S G_{S\pi}^2 (M_S^2 - M_\rho^2) = 0. \quad (20)$$

which can hold only if there is at least one S for which $M_S \leq M_\rho$. Hence chiral symmetry and VMD place an upper bound on the mass of the lowest-lying scalar in the hadron spectrum. The inequality generalizes to $M_S \leq M_\rho \tan \phi$ for arbitrary ϕ .

This argument relies on the assumption that the lowest-lying S state is in a chiral multiplet with the pion. A scalar in a separate multiplet, for instance in the $(\mathbf{1}, \mathbf{1})$ representation, would be decoupled from the pion and hence would not contribute to the low-energy constants of chiral perturbation theory.

5.4 Explicit Breaking Effects

Taking into account a nonvanishing pion mass, Eq. (11b) and Eq. (11e) become:

$$\sum_V F_V^2 M_V^2 - \sum_A F_A^2 M_A^2 = F_\pi^2 M_\pi^2 \quad (21a)$$

$$\sum_V G_{V\pi}^2 (M_\pi^2 - M_V^2) - \sum_S G_{S\pi}^2 (M_\pi^2 - M_S^2) = 0. \quad (21b)$$

In the simplest nontrivial scenario this leads to the modified mass relations,

$$M_\rho^2 = M_{a_1}^2 \cos^2 \phi + M_\pi^2 \sin^2 \phi \quad (22a)$$

$$(M_{f_0}^2 - M_\pi^2) = (M_\rho^2 - M_\pi^2) \tan^2 \phi, \quad (22b)$$

which imply (here we ignore the additional operators with explicit breaking in chiral perturbation theory at order p^4) the modifications:

$$l_1 \rightarrow l_1 - \bar{l} \frac{M_\pi^2}{2M_\rho^2} \cot^4 \phi \cos 2\phi \quad (23a)$$

$$l_5 \rightarrow l_5 + \bar{l} \frac{M_\pi^2}{4M_\rho^2} \cos^4 \phi. \quad (23b)$$

Amusingly, the corrections to l_1 vanish for the choice $\phi = 45^\circ$. And the corrections to l_5 are insignificant indeed:

$$l_5 = -\bar{l} \left\{ \frac{3}{8} - \left(\frac{M_\pi}{4M_\rho} \right)^2 \right\}. \quad (24)$$

These corrections are meant to be indicative of the size of explicit breaking effects. As is usual in chiral perturbation theory there are other effects arising at the next order (p^6) in the chiral expansion which further shift the l 's [16].

6. Conclusion

The full $SU(2) \times SU(2)$ chiral symmetry of QCD places significant constraints on resonance saturation in chiral perturbation theory. Although some of these constraints have been studied previously, here *all* of the constraints relevant to $\pi - \pi$ scattering have been taken into account. We have found that the simple picture with a single vector and a single axialvector state saturating the low-energy constants of chiral perturbation theory is inconsistent with chiral symmetry. This is easily seen by counting degrees of freedom and matching to the dimensionality of allowed chiral representations. In particular, it would seem that chiral symmetry requires the presence of isoscalar resonances. We have shown that the lightest scalar mass is bounded above by M_ρ if vector meson dominance is assumed.

According to the chiral symmetry point of view advocated here, vector meson dominance is a consequence of the fact that in QCD, the pion chiral representation is the lowest-dimensional nontrivial representation, the ten dimensional $(\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{2})$ representation, where the angle ϕ which mixes the $(\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{1})$ and $(\mathbf{2}, \mathbf{2})$ representations takes the value 45° . Why the $(\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{1})$ and $(\mathbf{2}, \mathbf{2})$ representations enter with equal weight is mysterious and has been investigated in Ref. 17 and Ref. 18.

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